

T top space, $T = \bigcup_{i \in I} U_i$ be an open covering.

Fix an well ordering on the indexing set I . Let $\mathcal{U}$ denote the ordered collection $\{U_i\}_{i \in I}$

Given a sheaf of ab grps $\mathcal{F}$, one can associate a complex of sheaves denoted $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ where

• For $p \geq 0$, $\underline{C}^p(\mathcal{U}, \mathcal{F})$

$$= \prod_{i_0 < i_1 < \dots < i_p} (\hat{\partial}_{i_0 \dots i_p})^* (\mathcal{F}|_{U_{i_0 \dots i_p}}) \quad \text{where } U_{i_0 \dots i_p} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p},$$

and $\hat{\partial}_{i_0 \dots i_p}: U_{i_0 \dots i_p} \hookrightarrow X$ is the inclusion.

• for $p < 0 = 0$

So for $V \subseteq T$,

$$\underline{C}^p(\mathcal{U}, \mathcal{F})(V) = \prod_{i_0 < i_1 < \dots < i_p} \mathcal{F}(V \cap U_{i_0 \dots i_p})$$

• For $\underline{C}^p(\mathcal{U}, \mathcal{F})(V) \ni \alpha = (\alpha_{i_0 \dots i_p})$

$d\alpha \in \underline{C}^{p+1}(\mathcal{U}, \mathcal{F})$ is given by

$$(d\alpha)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \dots i_{j-1} \hat{i}_j i_{j+1} \dots i_{p+1}}|_{U_{i_0 \dots i_{p+1}} \cap V}$$

of sheaves

Def: • $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ is called the Čech complex_n of $\mathcal{F}$ for the covering $\mathcal{U}$.

$\check{H}^i(\mathcal{U}, \mathcal{F}) := H^i(\underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$ called the i -th Čech cohomology sheaf

• $\underline{C}^\bullet(\mathcal{U}, \mathcal{F}) = \Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$

Čech complex of $\mathcal{F}$ for the covering $\mathcal{U}$.

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Ex: $X = \mathbb{P}^1_k = \text{Proj}(k[x, y])$ $\mathcal{U} = (D_+(x), D_+(y))$

$$C^\bullet(\mathcal{U}, \mathcal{O}_X) =$$

$$0 \rightarrow \Gamma(D_+(x), \mathcal{O}_X) \oplus \Gamma(D_+(y), \mathcal{O}_X) \rightarrow \Gamma(D_+(xy), \mathcal{O}_X) \rightarrow 0 \dots$$

$$0 \rightarrow k[x/y] \oplus k[x/y] \rightarrow k[x/y, y/x] \rightarrow 0$$

$$(\alpha_0, \alpha_1) \rightarrow \alpha_{01} = \alpha_1|_{D_+(xy)} - \alpha_0|_{D_+(xy)}$$

$$\check{H}^0(\mathcal{U}, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = k$$

$$\check{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$$

Lemma: With the notation as above, there is a natural

$$\text{map } \mathcal{F} \rightarrow \underline{C}^\bullet(\mathcal{U}, \mathcal{F}) \text{ sending}$$

$$\alpha \in \mathcal{F}(U) \text{ to } (\alpha|_{U_i \cap U_j})_{i,j}.$$

This map induces isom:

$$\mathcal{F} = \underline{H}^0(\mathcal{U}, \mathcal{F}) := H^0(\underline{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

$$\Gamma(T, \mathcal{F}) = \underline{\check{H}}^0(\mathcal{U}, \mathcal{F}) := H^0(\Gamma(T, \underline{C}^\bullet(\mathcal{U}, \mathcal{F})))$$

Thm: The map $\mathcal{F} \rightarrow \underline{C}^\bullet(\mathcal{U}, \mathcal{F})$ provided above

gives a resolution of $\mathcal{F}$ by $\underline{C}^\bullet(\mathcal{U}, \mathcal{F})$. That is

$$\textcircled{1} \underline{H}^i(\mathcal{U}, \mathcal{F}) = 0 \quad \forall i > 0, \quad \textcircled{2} \mathcal{F} \rightarrow \underline{H}^0(\mathcal{U}, \mathcal{F}) \text{ is an isom.}$$

Pf $\textcircled{2}$ is fine, for $\textcircled{1}$ see Hart, Lemma 4.2.

Thm: $T, \mathcal{U}$ be as above. For a flaque sheaf of $\mathcal{F}$,

$$H^i(\mathcal{U}, \mathcal{F}) = 0 \quad \forall i > 0.$$

Pf: Since $\mathcal{F}$ is flaque, each $\mathcal{F}|_{U_{i_0} \dots i_p}$ is flaque

$$\Rightarrow (j_{i_0 \dots i_p})_* (\mathcal{F}|_{U_{i_0 \dots i_p}}) \text{ is flaque}$$

$$\Rightarrow (j_{i_0 \dots i_p})_* (\mathcal{F}|_{U_{i_0 \dots i_p}}) \text{ is flasque}$$

$$\Rightarrow \underline{C}^p(U, \mathcal{F}) \text{ being fdt of flasque sheaves is flasque.}$$

Then $\underline{C}^\bullet(U, \mathcal{F})$ is a flasque resolution of $\mathcal{F}$.

Then $\Gamma(T, \underline{C}^\bullet(U, \mathcal{F})) = \underline{C}^\bullet(U, \mathcal{F})$ computes the sheaf cohomologies of $\mathcal{F}$.

Since $\mathcal{F}$ is flasque

$$0 = H^i(T, \mathcal{F}) = H^i(\underline{C}^\bullet(U, \mathcal{F})) = \check{H}^i(U, \mathcal{F}) \quad \forall i > 0.$$

Thm. 0 Let X be noetherian, separated scheme. Let $U = \{U_i\}_{i \in I}$ be a finite affine open covering. For a q-coh sheaf $\mathcal{F}$,

$$H^i(X, \mathcal{F}) \cong \check{H}^i(U, \mathcal{F})$$

② X, U as above, $f: X \rightarrow Y$ be a separated map

$$\text{Then } R^i f_* \mathcal{F} \cong H^i(f_* \underline{C}^\bullet(U, \mathcal{F}))$$

Pf. ① First we show that $\{\check{H}^i(U, -)\}_{i \in \mathbb{N}}$ is a universal δ -functor from $\mathcal{Q}coh(X) \rightarrow \mathcal{A}b_{gr}$.
and functoriality

• The additivity of each $\check{H}^i(U, -)$ is clear.

• We need to check that given a short exact seq

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \text{ in } \mathcal{Q}coh(X),$$

$\{\check{H}^i(U, -)\}_{i \in \mathbb{N}}$ form a long exact seq.

Note (i) Each $U_{i_0 \dots i_p}$ is affine as X is separated.

$$qcoh \Rightarrow 0 \rightarrow \mathcal{F}'|_{U_{i_0 \dots i_p}} \rightarrow \mathcal{F}|_{U_{i_0 \dots i_p}} \rightarrow \mathcal{F}''|_{U_{i_0 \dots i_p}} \rightarrow 0$$

is exact $\forall i_0 < i_1 < \dots < i_p, \forall p$.

$$\Rightarrow 0 \rightarrow \Gamma(X, \underline{C}^1(U, \mathcal{F}')) \rightarrow \Gamma(X, \underline{C}^1(U, \mathcal{F})) \rightarrow \Gamma(X, \underline{C}^1(U, \mathcal{F}'')) \rightarrow 0$$

is exact

$\Rightarrow$ we have a short exact seq of complexes

$$0 \rightarrow \underline{C}^\bullet(U, \mathcal{F}') \rightarrow \underline{C}^\bullet(U, \mathcal{F}) \rightarrow \underline{C}^\bullet(U, \mathcal{F}'') \rightarrow 0$$

This form a long exact seq of Čech cohomology grps.

Universality of $\{\check{H}^i(U, -)\}$ - It's enough to show

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That each $\check{H}^i(U, -)$ is effaceable in $\mathcal{Q}coh(X)$ for each $i > 0$
Given $\mathcal{F} \in \mathcal{Q}coh(X)$, choose an embedding

$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}$ where $\mathcal{I}$ is $q.coh$ flaque

Since $H^i(U, \mathcal{I}) = 0 \quad \forall i > 0$, we are done.

On the other hand $\{H^i(X, -)\}_{i \in \mathbb{N}}$ is a universal

δ -functor from $\mathcal{Q}coh(X) \rightarrow \mathbf{Ab}$

Universality follows since every object can
be embedding into a flaque $q.coh$ sheaf and
flaque objects are $\Gamma(X, -)$ acyclic.

Since $\check{H}^0(U, \mathcal{F}) \cong H^0(X, \mathcal{F})$ we are done by universality.

② Since f is separated, for any affine $^{\text{open}} V \subseteq Y$ and
 $U \subseteq X$,
 $f^{-1}(V) \cap U$ is affine.

Finish The rest as H.W (follows from ①).

End of 13.12.24
Lecture